International Journal of Scientific Engineering and Technology (ISSN : 2277-1581)
Volume No.3 Issue No.7, pp : 848-851 1 July 2014

Some New Exact Travelling Wave Solutions of the Cubic Nonlinear Schrodinger Equation using the $(\mathrm{Exp}(-\phi(\xi)))$ -Expansion Method

M. Golam Hafez, M. A. Kauser, M. Tahmina Akter
Department of Mathematics, Chittagong University of Engineering and Technology, Bangladesh
*Email: golam_hafez@yahoo.com

Abstract: ***The nonlinear physical model such as the cubic nonlinear Schrodinger equation has been applied in many branches of physics. In this paper, the*** $\exp(-\phi(\xi))$ ***-expansion method is applied to evaluate new exact travelling wave solutions of the complex Schrodinger equation with cubic nonlinearity. Various solutions of the cubic nonlinear Schrodinger equation using this method provide us the firm mathematical foundation of soliton physics. Further, three-dimensional plots of the solutions are also given to visualize the dynamics of the equation.***

Keywords: The $\exp(-\phi(\xi))$ **-expansion method; the cubic nonlinear Schrodinger equation; nonlinear evolution equations; travelling wave solutions; solitary wave solutions.**

Mathematics Subject Classification: 35K99, 35P05, 35P99.

1. Introduction

Nonlinear partial differential equations have been widely applied in many branches of mathematical physics and engineering such as fluids dynamics, solid mechanics, particle physics, optical fibers and plasma physics etc. Also, the nonlinear evaluation equations and the solitons concept have introduced remarkable achievements in the field of applied sciences and engineering. In particular, there has been considerable interest to evaluating more exact travelling wave solutions of the nonlinear evolution equations that describes some important physical and dynamic processes. Recently, many kinds of powerful methods have been proposed to find exact solutions of nonlinear PDEs e.g three-wave method [1], extended homoclinic test approach [2], the improved F-expansion method [3], the projective Riccati equation method [4], the Jacobi elliptic function expansion method [5, 6] and the tanh-function method [7-10]. For integrable nonlinear differential equations, the inverse scattering transform method [11], the Exp-function method [12-15], the extended tanh-method [16, 17], the homogeneous balance method [18-20] and the $\exp(-\varphi(\xi))$-expansion method [34, 35] etc are used for searching the exact solutions.

The paper is organized as follows: Section 2 presents the briefly description of the $\exp(-\phi(\xi))$ expansion method. Section 3 is devoted to derive the travelling wave solutions of the cubic nonlinear Schrodinger equation using the $\exp(-\phi(\xi))$ expansion method. The physical interpretation and graphical representations of the solutions are presented in section 4. The conclusion is given by in section 5.

2. The $\exp(-\phi(\xi))$ -expansion method

Let us consider a general nonlinear PDE in the form

$$F(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, \ldots), \tag{1}$$

where $u = u(x,t)$ is an unknown function, F is a polynomial in $u(x,t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives. The main steps of this method are as follows:

Step 1: Combine the real variables x and t by a complex variable ξ

$$u(x,t) = u(\xi),\ \xi = x \pm c\,t, \tag{2}$$

where c is the speed of the traveling wave. The traveling wave transformation (2) converts Eq. (1) into an ordinary differential equation (ODE) for $u = u(\xi)$:

$$\Re(u, u', u'', u''', \cdots), \tag{3}$$

where $\Re$ is a polynomial of u and its derivatives and the superscripts indicate the ordinary derivatives with respect to ξ.

Step 2. Suppose the traveling wave solution of Eq. (3) can be expressed as follows:

$$u(\xi) = \sum_{i=0}^{N} A_i \left(\exp(-\phi(\xi))\right)^i, \tag{4}$$

where A_i $(0 \le i \le N)$ are constants to be determined, such that $A_N \ne 0$ and $\phi = \phi(\xi)$ satisfies the following ordinary differential equation:

$$\phi'(\xi) = \exp(-\phi(\xi)) + \mu \exp(\phi(\xi)) + \lambda, \tag{5}$$

Eq. (5) gives the following solutions:

Family 1: When $\mu \ne 0,\ \lambda^2 - 4\mu > 0,$

$$\phi(\xi) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}(\xi + E)\right) - \lambda}{2\mu}\right) \tag{6}$$

Family 2: When $\mu \ne 0,\ \lambda^2 - 4\mu < 0,$

International Journal of Scientific Engineering and Technology (ISSN : 2277-1581)
Volume No.3 Issue No.7, pp : 848-851 1 July 2014

$$\phi(\xi)=\ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+E)\right)-\lambda}{2\mu}\right) \quad (7)$$

Family 3: When $\mu=0,\ \lambda\neq 0,$ and $\lambda^2-4\mu>0,$

$$\phi(\xi)=-\ln\left(\frac{\lambda}{\exp(\lambda(\xi+E))-1}\right) \quad (8)$$

Family 4: When $\mu\neq 0,\ \lambda\neq 0,$ and $\lambda^2-4\mu=0,$

$$\phi(\xi)=\ln\left(-\frac{2(\lambda(\xi+E)+2)}{\lambda^2(\xi+E)}\right) \quad (9)$$

Family 5: When $\mu=0,\ \lambda=0,$ and $\lambda^2-4\mu=0,$

$$\phi(\xi)=\ln(\xi+E) \quad (10)$$

$A_N,\cdots\cdots,c,\lambda,\mu$ are constants to be determined latter, $A_N\neq 0,$ the positive integer N can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (3).

Step 3: Substitute Eq. (4) into Eq. (3) and then we account the function $\exp(-\phi(\xi))$. As a result of this substitution, we get a polynomial of $\exp(-\phi(\xi))$. We equate all the coefficients of same power of $\exp(-\phi(\xi))$ to zero. This procedure yields a system of algebraic equations whichever can be solved to find $A_1,A_2\cdots\cdots,c,\lambda,\mu$ with the aid of Maple. Substituting the values of $A_1,A_2\cdots\cdots,c,\lambda,\mu$ into Eq. (4) along with general solutions of Eq. (5) completes the determination of the solution of Eq. (1).

3. Some new exact travelling wave solutions of Cubic nonlinear Schrodinger equation

Let us consider the eigenvalue problem [23],

$$\left.\begin{aligned}\frac{\partial}{\partial x}V_1+f(\lambda)V_1&=g(\lambda)\,q(x,t)V_2\\ \frac{\partial}{\partial x}V_2-f(\lambda)V_2&=g(\lambda)\,r(x,t)V_1\end{aligned}\right\} \quad (11)$$

where $f(\lambda)$ and $g(\lambda)$ are functions of the eigenvalue λ. Together equation (11), we assume that the eigenfunctions V_1 and V_2 evolve in time according to the temporal evolution equation,

$$\left.\begin{aligned}\frac{\partial}{\partial t}V_1&=A(\lambda,q,r)V_1+B(\lambda,q,r)V_2\\ \frac{\partial}{\partial t}V_2&=C(\lambda,q,r)V_1-A(\lambda,q,r)V_2\end{aligned}\right\} \quad (12)$$

where A, B and C depend on λ and the functional of the potentials q and r and of their spatial derivatives in the arbitrary order.

The soliton is to required that the eigenvalue λ does not change in time while the potential $q(x,t)$ and $r(x,t)$ change their shape in time. If we set, $\frac{\partial}{\partial t}\lambda=0$, then we find that A, B and C should satisfy the following set of equations,

$$\left.\begin{aligned}&\frac{\partial}{\partial x}A-g(\lambda)(r\,B-q\,C)=0\\ &g(\lambda)\frac{\partial}{\partial t}\,q-\frac{\partial}{\partial x}B-2f(\lambda)B-2g(\lambda)q\,A=0\\ &g(\lambda)\frac{\partial}{\partial t}\,r-\frac{\partial}{\partial x}C-2f(\lambda)C-2g(\lambda)r\,A=0\end{aligned}\right\} \quad (13)$$

For the above expression of $f(\lambda)$ and $g(\lambda)$, the functions A, B and C are constructed from (13) and obtain the nonlinear evolution equation for $q(x,t)$ and $r(x,t)$, which are the soliton equation.
If we choose $f(\lambda)=\lambda$, $g(\lambda)=1$ and $r=-\psi^*$ (complex), we obtain the cubic nonlinear Schrodinger equation [4],

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+|\psi|^2\psi=0 \quad (14)$$

To find out the travelling wave solutions of the cubic nonlinear Schrodinger, we introduce the following transformations

$$\psi(x,t)=\exp(i\,k)\times u(\xi),\ k=\alpha x+\beta t,\ \ \xi=x-c\,t \quad (15)$$

By substituting (15) into Eq. (14), we find that $c=2\alpha$ and u satisfy the following nonlinear ordinary differential equation:

$$\frac{d^2u}{d\xi^2}-(\beta+\alpha^2)u+u^3=0 \quad (16)$$

By balancing $\frac{d^2u}{d\xi^2}$ and u^3, the pole of the equation (16) is $N=1$. Therefore, the $\exp(-\phi(\xi))$-expansion method admits the solution of (16) in the form

$$u(\xi)=A_0+A_1\exp(-\phi(\xi)),\ \ A_1\neq 0 \quad (17)$$

Substitute (5) and (17) into the Eq. (16) and equating the coefficient of $(\exp(-\phi(\xi)))^i$, $(i=0,1,2,\cdots,6)$ are equal to zero, yielding a set of algebraic equations with the aid of Maple as follows:

$$-\alpha^2A_0+A_0^3+A_1\,\mu\,\lambda-\beta\,A_0=0 \quad (18)$$

$$3A_0^2\,A_1-\beta\,A_1-\alpha^2A_1+A_1\,\lambda^2+2\,A_1\mu=0 \quad (19)$$

$$3\,A_1\,\lambda+3A_0\,A_1^2=0 \quad (20)$$

$$2A_1+A_1^3=0 \quad (21)$$

Solving the above algebraic equations, we obtain a set of solution as follows:

$$\left\{\alpha=\alpha,\beta=-\alpha^2-\frac{\lambda^2}{2}+2\mu,A_0=\pm\lambda\frac{i}{\sqrt{2}},\ A_1=\pm i\sqrt{2},\ c=2\alpha\right\} \quad (22)$$

where α, λ and μ are arbitrary constants.
By substituting (22) into Eq. (17), we have

$$u(\xi)=\pm\frac{i\lambda}{\sqrt{2}}\pm i\sqrt{2}\exp(-\phi(\xi)) \quad (23)$$

where $\xi=x-2\alpha t.$

Again, by substituting (6), (7), (8), (9) and (10) into the equation (23) respectively, we have the travelling wave solutions of the equation (14) are as follows:

International Journal of Scientific Engineering and Technology (ISSN : 2277-1581)
Volume No.3 Issue No.7, pp : 848-851 1 July 2014

When $\lambda^2-4\mu>0, \mu\neq 0$, then

$$\psi_1(x,t)=\exp\left[i\left\{\alpha\, x+\left(-\alpha^2-\frac{\lambda^2}{2}+2\mu\right)t\right\}\right]\times \left\{\pm\frac{i\lambda}{\sqrt{2}}\mp i\sqrt{2}\left(\frac{2\mu}{\sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2}(x-2\alpha\, t+E)\right)+\lambda}\right)\right\}, \quad (24)$$

where $\Omega=\lambda^2-4\mu$ and E is arbitrary constant.

When $\lambda^2-4\mu<0$, then

$$\psi_2(x,t)=\exp\left[i\left\{\alpha\, x+\left(-\alpha^2-\frac{\lambda^2}{2}+2\mu\right)t\right\}\right]\times \left\{\pm\frac{i\lambda}{\sqrt{2}}\pm i\sqrt{2}\left(\frac{2\mu}{\sqrt{\Omega}\tan\left(\frac{\sqrt{\Omega}}{2}(x-2\alpha\, t+E)\right)-\lambda}\right)\right\}, \quad (25)$$

where $\Omega=4\mu-\lambda^2$ and E is arbitrary constant.

When $\lambda^2-4\mu>0, \mu=0$, then

$$\psi_3(x,t)=\exp\left[i\left\{\alpha\, x+\left(-\alpha^2-\frac{\lambda^2}{2}+2\mu\right)t\right\}\right]\times \left\{\pm\frac{i\lambda}{\sqrt{2}}\pm i\sqrt{2}\left(\frac{\lambda}{\exp(\lambda(x-2\alpha\, t+E))-1}\right)\right\}, \quad (26)$$

where E is arbitrary constant.

When $\lambda^2-4\mu=0, \mu\neq 0, \lambda\neq 0$, then

$$\psi_4(x,t)=\exp\left[i\left\{\alpha\, x+\left(-\alpha^2-\frac{\lambda^2}{2}+2\mu\right)t\right\}\right]\times \left\{\pm\frac{i\lambda}{\sqrt{2}}\mp i\sqrt{2}\left(\frac{\lambda^2(x-2\alpha\, t+E)}{\lambda(x-2\alpha\, t+E)+2}\right)\right\}, \quad (27)$$

where E is arbitrary constant.

When $\lambda^2-4\mu=0, \mu=0, \lambda=0$, then

$$\psi_5(x,t)=\exp\left[i\left\{\alpha\, x+\left(-\alpha^2-\frac{\lambda^2}{2}+2\mu\right)t\right\}\right]\times \left\{\pm\frac{i\lambda}{\sqrt{2}}\pm i\sqrt{2}\left(\frac{1}{x-2\alpha t+E}\right)\right\}, \quad (28)$$

4. Physical Interpretation

In this section, we describe the physical interpretation and graphical representation of the solutions of the cubic nonlinear Schrodinger equation.

4.1 Interpretations

The solution $\psi_1(x,t)$ of the equation (14) is represented the soliton solution. Solitons are very special kinds of solitary waves which described many physical phenomena in soliton physics. It has a remarkable property, that is, it keeps its identity upon interacting with other solitons. Soliton solutions also give rise to particle-like structures, such as magnetic monopoles etc. So, soliton are everywhere in the nature. The Fig. 1 has been shown the shape of the solution $\psi_1(x,t)$ for $\alpha=2,\ E=1,\ \mu=2,\ \lambda=3$ with $-10\le x,t\le 10$.

The solution $\psi_2(x,t)$ of the equation (14) is described the periodic soliton solution for various values of the physical parameters. The Fig. 2 has been shown the shape of the solution $\psi_2(x,t)$ for $\alpha=2,\ E=1,\ \mu=3,\ \lambda=1$ with $-10\le x,t\le 10$.

The solution $\psi_3(x,t)$ of the equation (14) is also presented the soliton solution which is shown in Fig. 3 for $\alpha=0.5,\ E=1,\ \mu=0,\ \lambda=2$ with $-10\le x,t\le 10$.

The Fig. 4 has been shown the shape of the solitary soliton solution of $\psi_4(x,t)$ for $\alpha=2,\ E=1,\ \mu=2,\ \lambda=3$ with $-10\le x,t\le 10$.

Finally, solution $\psi_5(x,t)$ is cuspon of the cubic nonlinear Schrodinger equation (14). Cuspons are other kinds of solitons where solution exhibits cusps at their crests. The Fig. 5 shows the shape of the coupson soliton solution, obtain from $\psi_5(x,t)$ for $E=1,\ \alpha=-0.5,\ \mu=0,\ \lambda=0$ with $-10\le x,t\le 10$.

4.2 Graphical representations

The graphical illustrations of the solutions are given below in the figures (Fig. 1-5) with the aid of Maple.

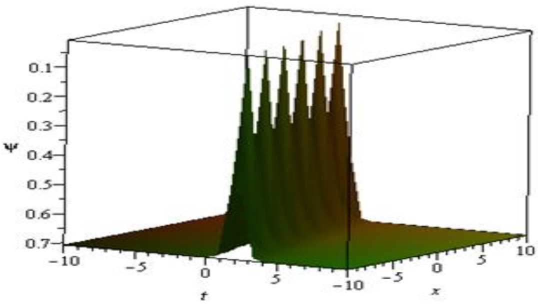

Fig. 1: Modulus plot of Soliton solution, Shape of Eq. (24) when $\alpha=2,\ E=1,\ \mu=2,\ \lambda=3$ with $-10\le x,t\le 10$.

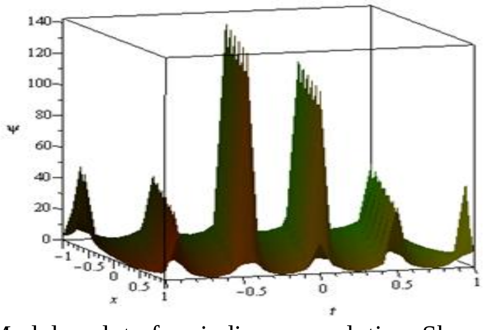

Fig. 2: Modulus plot of periodic wave solution, Shape of (25) when $\alpha=2,\ E=1,\ \mu=3,\ \lambda=1$ with $-10\le x,t\le 10$.

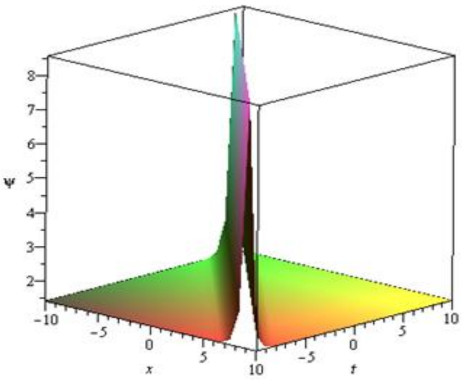

Fig. 3: Modulus plot of Soliton solution, Shape of Eq. (26) when $\alpha = 0.5,\ E = 1,\ \mu = 0,\ \lambda = 2$ with $-10 \le x, t \le 10$.

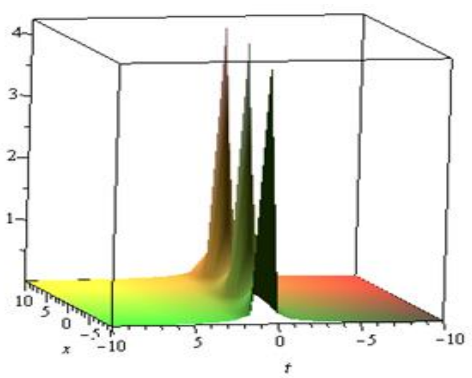

Fig. 4: Modulus plot of solitary Soliton solution, Shape of Eq. (27) when $\alpha = 2,\ E = 1,\ \mu = 2,\ \lambda = 3$ with $-10 \le x, t \le 10$.

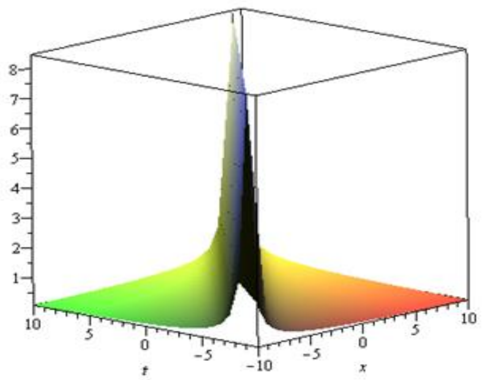

Fig. 5: Modulus plot of cuspon solution, Shape of (28) when $E = 1,\ \alpha = -0.5,\ \mu = 0,\ \lambda = 0$ with $-10 \le x, t \le 10$.

5. Conclusion

The $\exp(-\phi(\xi))$ -expansion method has been successfully applied to establish new travelling wave solutions to the cubic nonlinear Schrodinger equation. In present research work, we observe that various soliton phenomena have gives the mathematical foundation in soliton physics. The various types of soliton solutions are helpful to describe the soliton propagation in soliton physics, such as soliton propagation in optical fibers etc. The performance of this method is reliable, convincing and can be used to other NLEEs in finding exact solutions. Although the method has a lot of merit it has a few drawbacks, such as, sometimes the method gives solutions in disguised versions of known solutions that may be found by other methods.

References

i. MT Darvishi, Mohammad Najafi. Some exact solutions of the (2+1)-dimensional breaking soliton equation using the three-wave method, World Academy of Science, En-gineering and Technology. 2012, 87, 31-34.

ii. T Darvishi, Mohammad Najafi. Some complexiton type solutions of the (3+1)-dimensional Jimbo-Miwa equation, World Academy of Science, Engineering and Technology. 2012, 87, 42-44.

iii. Wang and HQ Zhang. Further improved F-expansion method and new exact solutions of Konopelchenko–Dubrovsky equation. Chaos, Solitons and Fract. 2005, 25. 601-610.

iv. Z Yan. Generalized method and its application in the higher-order nonlinear Schrodinger equation in nonlinear optical fibres. Chaos, Solitons and Fract. 2003, 16.759-766.

v. Y Chen and Q Wang. Extended Jacobi elliptic function rational expansion method and abundant families of Jacobi elliptic functions solutions to (1+1)-dimensional dispersive long wave equation. Chaos, Solitons and Fract. 2005, 24.745-757.

vi. S Liu, Z Fu, SD Liu and Q Zhao. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. 2001, A, 289. 69-74.

vii. W Malfliet.The tanh method: A tool for solving certain classes of nonlinear evolution and wave equations, J. Comput. Appl. Math. 2004, 164–165.529–541.

viii. W Malfliet. Solitary wave solutions of nonlinear wave equations, Am. J. Phys. 1992, 60 (7). 650–654.

ix. MA Abdou.The extended tanh-method and its applications for solving nonlinear physical models, Appl. Math. Comput. 2007, 190. 988-996.

x. AM Wazwaz.The extended tanh-method for new compact and non-compact solutions for the KP-BBM and the ZK-BBM equations. Chaos, Solitons and Fractals. 2008, 38. 1505-1516.

xi. MJ Ablowitz and P A Clarkson. Solitons, nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge Univ. Press, Cam-bridge.1991.

xii. JH He and XH Wu. Exp-function method for nonlinear wave equations. Chaos, Solitons and Fract. 2006, 1982, 30 (3). 700–708.

xiii. H Naher, FA Abdullah and MA Akbar. New traveling wave solutions of the higher dimensional nonlinear partial differential equation by the Exp-function method, J. Appl. Math. 2012, Vol. 2012, 14 pages. doi: 10.1155/2012/575387.

xiv. ST Mohyud-Din, MA Noor and A Waheed. Exp-function method for generalized travelling solutions of Calogero-Degasperis-Fokas equation. Zeitschrift für Naturforschung A. J. Phy. Sci. 2010, 65a. 78-84.

xv. MA Akbar and NHM Ali. New Solitary and Periodic Solutions of Nonlinear volution Equation by Exp-function Method, World Appl. Sci. J. 2012, Vol. 17(12). 1603-1610.

xvi. MA Abdou and AA Soliman. Modified extended tanh-function method and its application on nonlinear physical equations, Phys. Lett. A. 2006, 353 (6). 487–492.

xvii. SA El-Wakil and MA Abdou. New exact travelling wave solutions using modified extended tanh-function method. Chaos, Solitons and Fract. 2007, 31 (4). 840–852.

xviii. X Zhao, L Wang and W Sun. The repeated homogeneous balance method and its applications to nonlinear partial differential equations. Chaos, Solitons and Fractals. 2006, 28 (2). 448–453.

xix. F Zhaosheng.Comment on "On the extended applications of homogeneous balance method", Appl. Math. Comput. 2004, 158 (2). 593–596.

xx. X Zhao and D Tang. A new note on a homogeneous balance method, Phys. Lett. A. 2002, 297 (1–2). 59–67.

xxi. M.M. Zhao and C. Li, The $\exp(-\Phi(\eta))$ *-expansion method applied to nonlinear evolution equations, http://www. Paper. Edu. Cn*

xxii. K. Khan and M.A. Akbar, Application of $\exp(-\Phi(\eta))$ *-expansion method to find the exact solutions of modified Benjamin-Bona-Mahony equation, World Applied Sciences Journal, 24(10), 2013, 1373-1377.*

xxiii. Ablwitz, M. J. D. J. kaup, A. C. Newell and H. Segur, Stud. App. Math. 53, 1974, 249-315.