

# Pseudospherical Surfaces and Evolution Equations in Higher Dimensions

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**Abstract :** In this paper, the study of evolution equations with two independent variables which are related to pseudospherical surfaces in  $R^3$ , is extended to evolution equations with more than two independent variables. Equations of the type

$$u_{xt} = \psi(u, u_x, \dots, \frac{\partial^k u}{\partial x^k}, u_y, \dots, \frac{\partial^k u}{\partial y^k})$$

are studied and characterized. Some features and results on properties of these equations are given via this study.

**Keywords-**Evolution equations, Pseudospherical surfaces, Riemannian manifold and Solitons.

## I. Introduction

The study of non-linear evolution equations has been closely related to the study of soliton phenomena. In particular, many non-linear evolution equations of one spatial variable plus the time variable, which admit soliton solutions, have been extensively studied in the last two decades or so [v,ii]. Many interesting features of solitons, accordingly to evolution equations which admit, these soliton solutions, have been disclosed, [xi-vi,x,ii]. On the contrary, for the higher dimensional case, the studies of solitons are less developed and remain one of the interesting, and challenging, present and future research subjects, [vi,i]. This is also, the case for non-linear evolution equations with two or more spatial variables plus the time variable, [iv,xiii,xiv]. However, one of the main geometrical techniques, motivated in part, by Sasaki [x], El-Sabbagh [vi], Chern and Tenenblat [xii], is the notion of a differential equation which describes a pseudo spherical surface (P.S.S). With this concept, a systematic procedure has begun to obtain linear systems associated to the non-linear differential equations as well. These linear systems are essential in order to apply the inverse scattering method to obtain solutions of the non-linear differential equation, [viii,ix].

In this paper, we shall extend the notion of P.S.P to higher dimensions i.e. 3-dim plane of constant sectional curvature-1 imbedded in  $R^5$ . Conditions for equations of the type

$$u_{xt} = \psi(u, u_x, u_{xx}, \dots, \frac{\partial^k u}{\partial x^k}, u_y, u_{yy}, \dots, \frac{\partial^k u}{\partial y^k})$$

To describe a two-parameter 3-dim P.S.P, will be given in section III. While, in section II, we give basic notations and definitions as well as necessary preliminaries.

## II. Basic notations and Preliminaries

Let  $M$  be an  $n$ -dimensional Riemannian manifold with constant curvature, isometrically immersed in  $\bar{M}^{2n-1}$  with constant curvature  $\bar{K}$ , with  $K < \bar{K}$ . Let  $e_1, e_2, \dots, e_{2n-1}$  be a moving orthonormal frame on an open set of  $\bar{M}$ , so that at points

of  $M$ ,  $e_1, e_2, \dots, e_n$  are tangents to  $M$ . Let  $\omega_A$  be the dual orthonormal coframe and consider  $\omega_{AB}$  defined by

$$de_A = \sum_B \omega_{AB} e_B$$

The structure equations of  $\bar{M}$  are

$$d\omega_A = \sum_B \omega_B \wedge \omega_{BA}, \quad \omega_{AB} + \omega_{BA} = 0 \quad (2.1)$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \bar{K} \omega_A \wedge \omega_B \quad \text{with } 1 \leq A, B, C \leq 2n-1 \quad (2.2)$$

Restricting these forms to  $M$  we have  $\omega_\alpha = 0$ , so (2.1) gives with  $n+1 \leq \alpha, \beta, \gamma \leq 2n-1$  and  $1 \leq I, J, L \leq n$ ,

$$d\omega_\alpha = \sum_I \omega_I \wedge \omega_{I\alpha} = 0 \quad (2.3)$$

$$d\omega_I = \sum_J \omega_J \wedge \omega_{IJ} \quad (2.4)$$

from (2.2) we obtain, Gauss equation

$$d\omega_{IJ} = \sum_L \omega_{IL} \wedge \omega_{LJ} + \sum_\alpha \omega_{I\alpha} \wedge \omega_{\alpha J} - \bar{K} \omega_I \wedge \omega_J \quad (2.5)$$

and Codazzi equation

$$d\omega_{I\alpha} = \sum_A \omega_{IA} \wedge \omega_{A\alpha} \quad (2.6)$$

$M$  has constant sectional curvature  $K$  if and only if

$$\Omega_{IJ} = d\omega_{IJ} - \sum_L \omega_{IL} \wedge \omega_{LJ} = -K \omega_I \wedge \omega_J \quad (2.7)$$

$$\sum_\alpha \omega_{I\alpha} \wedge \omega_{\alpha J} = (\bar{K} - K) \omega_I \wedge \omega_J \quad (2.8)$$

Also, equation (2.2) implies that

$$d\Omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta} \quad \text{With } \Omega_{\alpha\beta} = \sum_I \omega_{\alpha I} \wedge \omega_{I\beta}$$

The forms  $\Omega_{\alpha\beta}$  give the normal curvature of  $M$  and  $I = \sum_I (\omega_I)^2$  is its first fundamental form.

For our purpose in this paper, we write these equations when  $\bar{M}$  is taken to be  $R^5$  and  $M$  is a 3-dimensional submanifold with constant sectional curvature  $K = -1$  (i.e. pseudo spherical 3-plane in  $R^5$ ).

The equations take the forms

$$\left. \begin{aligned} d\omega_1 &= \omega_4 \wedge \omega_2 + \omega_5 \wedge \omega_3 \\ d\omega_2 &= -\omega_4 \wedge \omega_1 + \omega_6 \wedge \omega_3 \\ d\omega_3 &= -\omega_5 \wedge \omega_1 - \omega_6 \wedge \omega_2 \\ d\omega_4 &= \omega_1 \wedge \omega_2 \\ d\omega_5 &= \omega_1 \wedge \omega_3 \\ d\omega_6 &= \omega_2 \wedge \omega_3 \end{aligned} \right\} \quad (2.9)$$

where we have written

$$\omega_4 = \omega_{12}\omega_5 = \omega_{13}, \text{ and}$$

$$\omega_6 = \omega_{23} \text{ with } \omega_{ij} = -\omega_{ji}, i, j = 1, 2, 3, \omega_{ii} = 0$$

We shall recall here the definition of a differential equation to describe a pseudospherical surface, introduced in [xii] and modify it in order to suit our purposes here.

Definition 1. A differential equation E-for a real function  $u(x, y, t)$  describes a 3-dimensional pseudospherical plane in  $R^5$  (simply p.s.p.) if it is the necessary and sufficient condition for the existence of differentiable functions  $f_{\alpha i}, 1 \leq \alpha \leq 6$  and  $1 \leq i \leq 3$ , depending on  $u$  and its derivatives, such that the 1-forms

$$\omega_\alpha = f_{\alpha 1} dx + f_{\alpha 2} dy + f_{\alpha 3} dt \quad (2.10)$$

satisfy the structure equations of a 3-plane of constant sectional curvature  $-1$  in  $R^5$  i.e. equations (2.9).

Definition 2. We shall define such 3-dimensional P.S.P to be a two-parameters 3-dimensional P.S.P  $f_{31} = f_{41} = \zeta$  and  $f_{22} = f_{42} = \xi$ , with  $\zeta$  and  $\xi$  constant parameters. In Fact, one can see that when  $u(x, y, t)$  is a generic solution of E, it provides a metric defined on an open subset of  $R^3$ , whose sectional curvature is  $-1$  and the lengths of the vector fields  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial y}$  satisfy  $|\frac{\partial}{\partial x}|^2 \geq \zeta^2$ ,  $|\frac{\partial}{\partial y}|^2 \geq \xi^2$ .

III. Equations of type  $u_{xt} = \psi(u, u_x, \dots, \frac{\partial^k u}{\partial x^k}, u_y, \dots, \frac{\partial^{k'} u}{\partial y^{k'}})$

To study equation (1.1). we first write

$$z_0 = u, z_1 = u_x, z_2 = u_{xx}, \dots, z_k = \frac{\partial^k u}{\partial x^k} \text{ and}$$

$$z_{1'} = u_y, z_{2'} = u_{yy}, \dots, z_{k'} = \frac{\partial^{k'} u}{\partial y^{k'}} \text{ thus equation (1.1)}$$

becomes

$$z_{1,t} = \psi(z_0, z_1, \dots, z_k, z_{1'}, \dots, z_{k'}) \quad (3.1)$$

We shall consider equation (3.1) with the following assumptions

$$\left. \begin{aligned} z_{i,t} &= z_{i',t} = 0 \\ z_{i',x} &= z_{i,y} = 0 \\ z_{0,t} &= 0 \text{ for } 2 \leq i \leq k, 1 \leq i' \leq k' \end{aligned} \right\} \quad (3.2)$$

where the comma denotes partial differentiation with respect to the shown variable. Now consider the following ideal  $I$  of forms on the space of variables

$$\left. \begin{aligned} \Omega_i &= dz_i \wedge dt - z_{i+1} dx \wedge dt, \quad 0 \leq i \leq k-1 \\ \Omega_{i'} &= dz_{i'} \wedge dt - z_{i'+1} dy \wedge dt, \quad 0 \leq i' \leq k'-1 \\ \Omega_k &= dz_k \wedge dx \wedge dt + dz_{k'} \wedge dy \wedge dt \\ \Omega &= dz_i \wedge dx \wedge dy - \psi dx \wedge dy \wedge dt \end{aligned} \right\} \quad (3.3)$$

Note that assumptions (3.2) mean that  $u$  has no  $(xy)$  terms. Now, if we apply Cartan-Kahler theory, [xii] for equation (3.1) and using the notation above we can obtain the following result which relates solutions of the differential equation (1.1) with integral manifolds of the ideal  $I$  formed by the forms in (3.3).

Proposition 3.1

The ideal  $I$  is a closed differential ideal. Moreover, if  $u(x, y, t)$  is a solution of equation (1.1), then with the given notations, the map

$$\phi(x, y, t) = (x, y, t, z_0(x, y, t), z_1(x, y, t), \dots, z_k(x, y, t), z_{1'}(x, y, t), \dots, z_{k'}(x, y, t)) \quad (3.4)$$

defines an integral manifold of  $I$  Conversely, any 3-dimensional integral manifold of  $I$  given by

$$\phi(a, b, c) = (x(a, b, c), y(a, b, c), t(a, b, c), z_0(a, b, c), \dots, z_{k'}(a, b, c))$$

with  $dx, dy$  and  $dt$  are linearly independent, determines a local solution of equation (3.1).

Proof:

It is easy to show that  $I$  is a closed differential ideal, where  $d\Omega_i = \Omega_{i+1} \wedge dx \in I, d\Omega_{i'} = \Omega_{i'+1} \wedge dy \in I$   
 $d\Omega_k = 0 \in I, d\Omega = -d\psi \wedge dx \wedge dy \wedge dt$  and also  
 $d\Omega = -\psi_{z_0} dz_0 \wedge dx \wedge dy \wedge dt - \psi_{z_1} dz_1 \wedge dx \wedge dy \wedge dt - \dots$   
 $- \psi_{z_k} dz_k \wedge dx \wedge dy \wedge dt - \psi_{z_{1'}} dz_{1'} \wedge dx \wedge dy \wedge dt - \dots$   
 $- \psi_{z_{k'}} dz_{k'} \wedge dx \wedge dy \wedge dt$

From equation (3.3) we have

$$d\Omega = -\psi_{z_0} dx \wedge dy \wedge \Omega_0 - \psi_{z_1} dx \wedge dy \wedge \Omega_1 - \dots - \psi_{z_k} dy \wedge \Omega_k - \psi_{z_{1'}} dx \wedge dy \wedge \Omega_{1'} - \dots - \psi_{z_{k'}} dx \wedge \Omega_{k'} \in I$$

thend  $d\Omega \in I$

Also, suppose  $u(x, y, t)$  is a solution of (1.1) we need to show that for  $\phi$  defined by (3.4), we have  $\phi^* I = 0$ , where  $\phi^*$  is the pullback map of  $\phi$ . From the definition of  $\phi$  one can easily see that

$$\begin{aligned} \phi^* \Omega_k &= 0, \quad \phi^* \Omega = 0 \\ \phi^* \Omega_{i'} &= 0, \quad \phi^* \Omega_{i'} = 0 \text{ then } \phi^* I = 0 \end{aligned}$$

Conversely, sectioning these forms in  $I$ , one can show that the map

$$\phi: (a, b, c) \rightarrow (x(a, b, c), y(a, b, c), t(a, b, c), z_0(a, b, c), \dots, z_{k'}(a, b, c))$$

is a solution of equation (3.1). i.e. if this map  $\phi$  is an integral manifold of  $I$  such that  $dx \wedge dy \wedge dt \neq 0$ , then we locally have  $(a, b, c) = g(x, y, t)$ .

Taking  $\phi$  as  $\phi = \phi \circ g$  we get  $\phi^* \Omega_i = g^* \phi^* \Omega_i = 0$ ,  $\phi^* \Omega_{i'} = g^* \phi^* \Omega_{i'} = 0$

Similarly  $\phi^* \Omega_k = 0$  and  $\phi^* \Omega = 0$  so we can write

$$\begin{aligned} dz_i \wedge dt - z_{i+1} dx \wedge dt &= 0, \quad 0 \leq i \leq k-1 \\ dz_{i'} \wedge dt - z_{i'+1} dy \wedge dt &= 0, \quad 1 \leq i' \leq k'-1 \end{aligned}$$

and  $(-z_{1,t} + \psi(x, y, t, z_0, z_1, \dots, z_k, z_{1'}, \dots, z_{k'})) dx \wedge dy \wedge dt = 0$

So, with  $z_0 = u, z_1 = u_x, \dots, z_k = \frac{\partial^k u}{\partial x^k}, z_{1'} = u_y, \dots, z_{k'} = \frac{\partial^{k'} u}{\partial y^{k'}}$

the function  $u(x, y, t)$  is a solution of equation (1.1).

Now to characterize equation (1.1), we first give the following result:

LEMMA 3.1

Let  $z_{1,t} = \psi(z_0, z_1, \dots, z_k, z_{1'}, \dots, z_{k'})$ , be a differential equation which describe an  $(\zeta, \xi)$  3-dimensional P.S.P with the associated 1-forms  $\omega_\alpha = f_{\alpha 1} dx + f_{\alpha 2} dy + f_{\alpha 3} dt, \alpha = 1, 2, \dots, 6$  where  $f_{\alpha i}$  and  $\psi$  are real differentiable ( $C^\infty$ ) functions defined on an open connected subset  $U \subset R^{k+k'+1}$  with no explicit dependence on  $x, y$  and  $t$ . Then

$$\begin{aligned}
 & f_{11,z_i} = f_{12,z_i} = f_{21,z_i} = f_{22,z_i} = f_{51,z_i} \\
 & = f_{52,z_i} = f_{61,z_i} = f_{62,z_i} = 0 \quad \forall i \neq 1 \\
 & f_{11,z_{k'}} = f_{13,z_{k'}} = f_{21,z_{k'}} = f_{23,z_{k'}} = f_{33,z_{k'}} = f_{43,z_{k'}} \\
 & = f_{51,z_{k'}} = f_{53,z_{k'}} = f_{61,z_{k'}} = f_{63,z_{k'}} = 0 \\
 & f_{33,z_{k'-1}} = f_{43,z_{k'-1}} = 0 \\
 & f_{12,z_k} = f_{13,z_k} = f_{22,z_k} = f_{23,z_k} = f_{33,z_k} = f_{43,z_k} = \\
 & f_{52,z_k} = f_{53,z_k} = f_{62,z_k} = f_{63,z_k} = 0 \\
 & f_{33,z_{k-1}} = f_{43,z_{k-1}} = 0 \\
 & f_{11,z_1}^2 + f_{12,z_1}^2 + f_{21,z_1}^2 + f_{22,z_1}^2 + f_{51,z_1}^2 \\
 & + f_{52,z_1}^2 + f_{61,z_1}^2 + f_{62,z_1}^2 \neq 0
 \end{aligned}$$

In U, and

$$\begin{aligned}
 & - \sum_{i=0}^{k'-1} z_{i+1} f_{11,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{12,z_i} \\
 & = \zeta f_{22} - \xi f_{21} + \xi f_{51} + \zeta f_{52} \quad (3.6)
 \end{aligned}$$

$$\begin{aligned}
 & -\psi f_{11,z_1} + \sum_{i=0}^{k-1} z_{i+1} f_{13,z_i} \\
 & = \zeta f_{23} - f_{43} f_{21} + f_{51} f_{33} - \zeta f_{53} \quad (3.7)
 \end{aligned}$$

$$\begin{aligned}
 & -\psi f_{12,z_1} + \sum_{i=0}^{k'-1} z_{i+1} f_{13,z_{i'}} \\
 & = \xi f_{23} - f_{43} f_{22} + f_{52} f_{33} - \xi f_{53} \quad (3.8)
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{i=0}^{k'-1} z_{i+1} f_{21,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{22,z_i} \\
 & = \xi f_{11} - \zeta f_{12} + \xi f_{61} \\
 & - \zeta f_{62} \quad (3.9)
 \end{aligned}$$

$$\begin{aligned}
 & -\psi f_{21,z_1} + \sum_{i=0}^{k-1} z_{i+1} f_{23,z_i} = f_{11} f_{43} - \zeta f_{13} + f_{61} f_{33} \\
 & - \zeta f_{63} \quad (3.10)
 \end{aligned}$$

$$\begin{aligned}
 & -\psi f_{22,z_1} + \sum_{i=0}^{k'-1} z_{i+1} f_{23,z_{i'}} = f_{12} f_{43} - \zeta f_{13} + f_{62} f_{33} \\
 & - \zeta f_{63} \quad (3.11)
 \end{aligned}$$

$$f_{11} f_{52} - f_{12} f_{51} = f_{22} f_{61} - f_{21} f_{62} \quad (3.12)$$

$$\begin{aligned}
 & \sum_{i=0}^{k-2} z_{i+1} f_{33,z_i} = f_{11} f_{53} - f_{13} f_{51} + f_{21} f_{63} \\
 & - f_{23} f_{61} \quad (3.13)
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{i=0}^{k'-2} z_{i+1} f_{33,z_{i'}} = f_{12} f_{53} - f_{13} f_{52} + f_{22} f_{63} \\
 & - f_{23} f_{62} \quad (3.14)
 \end{aligned}$$

$$\begin{aligned}
 & f_{11} f_{22} - f_{12} f_{21} \\
 & = 0 \quad (3.15)
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{i=0}^{k-2} z_{i+1} f_{43,z_i} \\
 & = f_{11} f_{23} - f_{13} f_{21} \quad (3.16)
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{i=0}^{k'-2} z_{i+1} f_{43,z_{i'}} \\
 & = f_{12} f_{23} - f_{13} f_{22} \quad (3.17)
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{i=0}^{k'-1} z_{i+1} f_{51,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{52,z_i} \\
 & = \xi f_{11} - \zeta f_{12} \quad (3.18)
 \end{aligned}$$

$$\begin{aligned}
 & -\psi f_{51,z_1} + \sum_{i=0}^{k-1} z_{i+1} f_{53,z_i} \\
 & = f_{11} f_{33} - \zeta f_{13} \quad (3.19)
 \end{aligned}$$

$$\begin{aligned}
 & -\psi f_{52,z_1} + \sum_{i=0}^{k'-1} z_{i+1} f_{53,z_{i'}} \\
 & = f_{12} f_{33} - \xi f_{13} \quad (3.20)
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{i=0}^{k'-1} z_{i+1} f_{61,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{62,z_i} \\
 & = f_{21} f_{32} - f_{22} f_{31} \quad (3.21)
 \end{aligned}$$

$$\begin{aligned}
 & -\psi f_{61,z_1} + \sum_{i=0}^{k-1} z_{i+1} f_{63,z_i} \\
 & = f_{21} f_{33} - \zeta f_{23} \quad (3.22)
 \end{aligned}$$

$$\begin{aligned}
 & -\psi f_{62,z_1} + \sum_{i=0}^{k'-1} z_{i+1} f_{63,z_{i'}} \\
 & = f_{22} f_{33} - \xi f_{23} \quad (3.23)
 \end{aligned}$$

with the assumptions (3.2).

Proof:

In the space of variables  $(x, y, t, z_0, z_1, \dots, z_k, z_1, \dots, z_{k'})$  we consider the ideal  $I$  generated by  $\Omega_i, \Omega_{i'}, \Omega_k$ , and  $\Omega$  defined by equations (3.3) with  $\psi$  given by equation(3.1) It follows from proposition (3.1) that  $\Omega_i = \Omega_{i'} = \Omega_k = \Omega = 0$ , when restricted to each integral manifold of  $E$ .

Hence, for  $z_0, z_1, \dots, z_k, z_1, \dots, z_{k'}$  satisfying (3.1), we have

$$\left. \begin{aligned}
 dz_i \wedge dt &= z_{i+1} dx \wedge dt, & i &= 0, 1, \dots, k-1 \\
 dz_{i'} \wedge dt &= z_{i'+1} dy \wedge dt, & i' &= 0, 1, \dots, k'-1 \\
 dz_i \wedge dx \wedge dy - \psi dx \wedge dy \wedge dt &= 0
 \end{aligned} \right\} (3.24)$$

At the beginning by using assumptions (3.2) and (3.5) we have The 1-forms  $\omega_\alpha$  satisfy the structure equations (2.9) therefore,

$$\begin{aligned}
 & \sum_{i=0}^k f_{11,z_i} dz_i \wedge dx + \sum_{i=0}^{k'} f_{11,z_{i'}} dz_{i'} \wedge dx + \sum_{i=0}^k f_{12,z_i} dz_i \wedge dy \\
 & + \sum_{i'=0}^{k'} f_{12,z_{i'}} dz_{i'} \wedge dy + \sum_{i=0}^k f_{13,z_i} dz_i \wedge dt + \sum_{i'=0}^{k'} f_{13,z_{i'}} dz_{i'} \wedge dt \\
 & = (\zeta f_{22} - \xi f_{21} + \xi f_{51} - \zeta f_{52}) dx \wedge dy \\
 & + (\zeta f_{23} - f_{43} f_{21} + f_{51} f_{33} - \zeta f_{53}) dx \wedge dt \\
 & + (\xi f_{23} - f_{43} f_{22} + f_{52} f_{33} - \xi f_{53}) dy \wedge dt (*)
 \end{aligned}$$

From the above equation(\*) we can obtain equation(3.6),(3.7) and (3.8) by simple calculations and by using equations (3.24)

In similar way by using assumptions (3.2) and (3.5) we have the 1-forms  $\omega_\alpha$  satisfy the structure equations (2.9) then

$$\begin{aligned}
 & \sum_{i=0}^k f_{21,z_i} dz_i \wedge dx + \sum_{i'=0}^{k'} f_{21,z_{i'}} dz_{i'} \wedge dx + \sum_{i=0}^k f_{22,z_i} dz_i \wedge dy \\
 & + \sum_{i'=0}^{k'} f_{22,z_{i'}} dz_{i'} \wedge dy + \sum_{i=0}^k f_{23,z_i} dz_i \wedge dt + \sum_{i'=0}^{k'} f_{23,z_{i'}} dz_{i'} \wedge dt
 \end{aligned}$$

$$\begin{aligned}
 &= (-\zeta f_{12} + \xi f_{11} + \xi f_{61} - \zeta f_{62}) dx \wedge dy \\
 &\quad + (-\zeta f_{13} + f_{43} f_{11} + f_{61} f_{33} - \zeta f_{63}) dx \wedge dt \\
 &\quad + (-\xi f_{13} + f_{43} f_{12} + f_{62} f_{33} - \xi f_{63}) dy \\
 &\quad \wedge dt \quad (**)
 \end{aligned}$$

From the above equation(\*\*) we can obtain equation(3.9),(3.10) and (3.11) by simple calculations and by using equations (3.24) In similar way by using assumptions (3.2) and (3.5) we have the 1-forms  $\omega_\alpha$  satisfy the structure equations (2.9) then

$$\begin{aligned}
 &\sum_{i=0}^k f_{31,z_i} dz_i \wedge dx + \sum_{i'=0}^{k'} f_{31,z_{i'}} dz_{i'} \wedge dx + \sum_{i=0}^k f_{32,z_i} dz_i \wedge dy \\
 &+ \sum_{i'=0}^{k'} f_{32,z_{i'}} dz_{i'} \wedge dy + \sum_{i=0}^k f_{33,z_i} dz_i \wedge dt + \sum_{i'=0}^{k'} f_{33,z_{i'}} dz_{i'} \wedge dt \\
 &= (-f_{51} f_{12} + f_{52} f_{11} - f_{61} f_{22} + f_{62} f_{21}) dx \wedge dy \\
 &\quad + (-f_{52} f_{13} + f_{53} f_{12} - f_{62} f_{23} + f_{22} f_{63}) dy \\
 &\quad \wedge dt \\
 &\quad + (-f_{51} f_{13} + f_{53} f_{11} - f_{61} f_{23} - f_{63} f_{21}) dx \wedge dt \quad (***)
 \end{aligned}$$

From the above equation(\*\*\*) we can obtain equation(3.12),(3.13) and (3.14) by simple calculations and by using equations (3.24)

Similarly by using assumptions (3.2) and (3.5) we have the 1-forms  $\omega_\alpha$  satisfy the structure equations (2.9) then

$$\begin{aligned}
 &\sum_{i=0}^k f_{41,z_i} dz_i \wedge dx + \sum_{i'=0}^{k'} f_{41,z_{i'}} dz_{i'} \wedge dx + \sum_{i=0}^k f_{42,z_i} dz_i \wedge dy \\
 &+ \sum_{i'=0}^{k'} f_{42,z_{i'}} dz_{i'} \wedge dy + \sum_{i=0}^k f_{43,z_i} dz_i \wedge dt + \sum_{i'=0}^{k'} f_{43,z_{i'}} dz_{i'} \wedge dt \\
 &= (f_{11} f_{22} - f_{12} f_{21}) dx \wedge dy + (f_{12} f_{23} - f_{13} f_{22}) dy \wedge dt \\
 &\quad + (f_{11} f_{23} - f_{13} f_{21}) dx \wedge dt \quad (.)
 \end{aligned}$$

From the above equation(.) we can obtain equation(3.15),(3.16) and (3.17) by simple calculations and by using equations (3.24)

Similarly by using assumptions (3.2) and (3.5) we have the 1-forms  $\omega_\alpha$  satisfy the structure equations (2.9) then

$$\begin{aligned}
 &\sum_{i=0}^k f_{51,z_i} dz_i \wedge dx + \sum_{i'=0}^{k'} f_{51,z_{i'}} dz_{i'} \wedge dx + \sum_{i=0}^k f_{52,z_i} dz_i \wedge dy \\
 &+ \sum_{i'=0}^{k'} f_{52,z_{i'}} dz_{i'} \wedge dy + \sum_{i=0}^k f_{53,z_i} dz_i \wedge dt + \sum_{i'=0}^{k'} f_{53,z_{i'}} dz_{i'} \wedge dt \\
 &= (f_{11} f_{32} - f_{12} f_{31}) dx \wedge dy + (f_{12} f_{33} - f_{13} f_{32}) dy \wedge dt \\
 &\quad + (f_{11} f_{33} - f_{13} f_{31}) dx \wedge dt \quad (..)
 \end{aligned}$$

From the above equation(..) we can obtain equation(3.18),(3.19) and (3.20) by simple calculations and by using equations (3.24)

Finally by using assumptions (3.2) and (3.5) we have the 1-forms  $\omega_\alpha$  satisfy the structure equations (2.9) then

$$\begin{aligned}
 &\sum_{i=0}^k f_{61,z_i} dz_i \wedge dx + \sum_{i'=0}^{k'} f_{61,z_{i'}} dz_{i'} \wedge dx + \sum_{i=0}^k f_{62,z_i} dz_i \wedge dy \\
 &+ \sum_{i'=0}^{k'} f_{62,z_{i'}} dz_{i'} \wedge dy + \sum_{i=0}^k f_{63,z_i} dz_i \wedge dt + \sum_{i'=0}^{k'} f_{63,z_{i'}} dz_{i'} \wedge dt \\
 &= (f_{21} f_{32} - f_{22} f_{31}) dx \wedge dy + (f_{22} f_{33} - f_{23} f_{32}) dy \wedge dt \\
 &\quad + (f_{21} f_{33} - f_{23} f_{31}) dx \wedge dt \quad (..)
 \end{aligned}$$

From the above equation(..) we can obtain equation(3.21),(3.22) and (3.23) by simple calculations and by using equations (3.24)

Now by taking the  $z_k$  derivative of equations (3.13) and (3.16), and the  $z_{k'}$  derivative of equations (3.14) and (3.17), we

obtain by using equations

$$(3.5), f_{33,z_{k-1}} = f_{43,z_{k-1}} = 0; f_{33,z_{k'-1}} = f_{43,z_{k'-1}} = 0$$

Hence we have obtained the relations (3.5) and relations from (3.6)  $\rightarrow$  (3.23). Finally, we observe that

if  $f_{51,z_1}, f_{52,z_1}, f_{61,z_1}$  and  $f_{62,z_1}$  vanish simultaneously, then the equation(3.1) cannot be the necessary and sufficient condition for  $\omega_\alpha$  to satisfy the structure equations (2.9)

for a 3-dimensional P.S.P. This completes the proof of the  $\square$  lemma.

Now, based on the above lemma, we will try to formulate the following in a simple form. So, we introduce these notations

$$\left. \begin{aligned}
 L_1 &= f_{11} f_{51,z_1} - f_{51} f_{11,z_1}, & L_2 &= f_{21} f_{61,z_1} - f_{61} f_{21,z_1} \\
 H_1 &= f_{11} f_{11,z_1} - f_{51} f_{51,z_1}, & H_2 &= f_{21} f_{21,z_1} - f_{61} f_{61,z_1} \\
 P_1 &= f_{11,z_1} f_{51,z_1} - f_{51,z_1} f_{11,z_1}, & P_2 &= f_{21,z_1} f_{61,z_1} - f_{61,z_1} f_{21,z_1} \\
 M_1 &= f_{51,z_1}^2 - f_{11,z_1}^2, & M_2 &= f_{61,z_1}^2 - f_{21,z_1}^2
 \end{aligned} \right\} (3.25)$$

and

$$A_1 = \sum_{i=0}^{k-2} z_{i+1} f_{33,z_i},$$

$$\begin{aligned}
 A_2 &= \sum_{i=0}^{k-2} z_{i+1} f_{43,z_i} \quad (3.26)
 \end{aligned}$$

Also, we consider the following

$$\left. \begin{aligned}
 L'_1 &= f_{12} f_{52,z_1} - f_{52} f_{12,z_1}, & L'_2 &= f_{22} f_{62,z_1} - f_{62} f_{22,z_1} \\
 H'_1 &= f_{12} f_{12,z_1} - f_{52} f_{52,z_1}, & H'_2 &= f_{22} f_{22,z_1} - f_{62} f_{62,z_1} \\
 P'_1 &= f_{12,z_1} f_{52,z_1} - f_{52,z_1} f_{12,z_1}, & P'_2 &= f_{22,z_1} f_{62,z_1} - f_{62,z_1} f_{22,z_1} \\
 M'_1 &= f_{52,z_1}^2 - f_{12,z_1}^2, & M'_2 &= f_{62,z_1}^2 - f_{22,z_1}^2
 \end{aligned} \right\} (3.27)$$

And

$$\begin{aligned}
 B_1 &= \sum_{i'=0}^{k'-2} z_{i'+1} f_{33,z_{i'}} \quad , \quad B_2 \\
 &= \sum_{i'=0}^{k'-2} z_{i'+1} f_{43,z_{i'}} \quad (3.28)
 \end{aligned}$$

Moreover whenever  $L_1 \neq 0, L_2 \neq 0, L'_1 \neq 0$  and  $L'_2 \neq 0$ , we define  $R^j$  and  $R^{j'}$  respectively as follows:  $R^{k-1} = 0$

$$\begin{aligned}
 R^j &= - \sum_{i=0}^{k-1} z_{i+1} R_{z_i}^{j+1} + f_{33,z_{j+1}} (H_1 + H_2) + f_{43,z_{j+1}} (H_1 + H_2) \\
 &\quad + \frac{R^{j+1}}{L_1} (z_2 L_{1,z_1} + \zeta H_1) + \frac{R^{j+1}}{L_2} (z_2 L_{2,z_1} + \zeta H_2) \\
 &\quad + \frac{1}{L_1} (A_1)_{z_{j+1}} (\zeta M_1 - z_2 P_1) \\
 &\quad + \frac{1}{L_2} (A_2)_{z_{j+1}} (\zeta M_2 - z_2 P_2) \quad (3.29)
 \end{aligned}$$

Where  $1 \leq j \leq k-2$  and  $R^{k-1} = 0$

$$\begin{aligned}
 R^{j'} &= - \sum_{i'=0}^{k'-1} z_{i'+1} R_{z_{i'}}^{j'+1} + f_{33,z_{j'+1}} (H'_1 + H'_2) \\
 &\quad + f_{43,z_{j'+1}} (H'_1 + H'_2) \\
 &\quad + \frac{R^{j'+1}}{L_1} (z'_2 L'_{1,z'_1} + \zeta H'_1) + \frac{R^{j'+1}}{L_2} (z'_2 L'_{2,z'_1} + \zeta H'_2)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{L_1} (B_1)_{z_{j'+1}} (\zeta M'_1 - z_2' P'_1) \\
 & + \frac{1}{L_2} (B_2)_{z_{j'+1}} (\zeta M'_2 - z_2' P'_2) \quad (3.30)
 \end{aligned}$$

Where  $1 \leq j' \leq k' - 2$ . Thus we have this theorem

**Theorem 3.1**

Let  $f_{\alpha 1}, f_{\alpha 2}$  and  $f_{\alpha 3}, 1 \leq \alpha \leq 6$  be differentiable functions of  $z_0, z_1, z_1', \dots, z_k, z_k'$  such that the relations (3.5) hold and  $f_{31} = f_{41} = \zeta, f_{32} = f_{42} = \xi$  two non zero parameters. Suppose  $H_1 L_1 \neq 0, H_2 L_2 \neq 0, H_1 L'_1 \neq 0$  and  $H'_2 L'_2 \neq 0$ . Then the equation  $z_{1,t} = \psi(z_0, z_1, z_1', \dots, z_k, z_k')$  describes a two-parameters 3-dimensional P.S.P with associated 1-forms

$$\omega_\alpha = f_{\alpha 1} dx + f_{\alpha 2} dy + f_{\alpha 3} dt, \quad 1 \leq \alpha \leq 6$$

If and only if the function  $\psi$  is given by

$$\begin{aligned}
 \psi = & \left( \frac{1}{L_1} + \frac{1}{L_2} \right) + \sum_{i=0}^{k-1} z_{i+1} (A_1)_{z_i} + \left( \frac{1}{L'_1} + \frac{1}{L'_2} \right) \\
 & + \sum_{i=0}^{k'-1} z'_{i+1} (B_1)_{z'_i} \\
 & + \frac{1}{H_1 L_1} \left( -z_2 \frac{L_1}{\zeta} + f_{51}^2 - f_{11}^2 \right) \sum_{i=0}^{k-2} z_{i+1} R^i \\
 & + \frac{1}{H_2 L_2} \left( -z_2 \frac{L_2}{\zeta} + f_{61}^2 - f_{21}^2 \right) \sum_{i=0}^{k-2} z_{i+1} R^i \\
 & + \frac{1}{H'_1 L'_1} \left( -z'_2 \frac{L'_1}{\xi} + f_{52}^2 - f_{12}^2 \right) \sum_{i=0}^{k'-2} z'_{i+1} R^{i'} \\
 & + \frac{1}{H'_2 L'_2} \left( -z'_2 \frac{L'_2}{\xi} + f_{62}^2 - f_{22}^2 \right) \sum_{i=0}^{k'-2} z'_{i+1} R^{i'} \\
 & + \frac{A_1}{H_1 L_1} (z_2 M_1 + \zeta L_1) + \frac{A_1}{H_2 L_2} (z_2 M_2 + \zeta L_2) \\
 & + \frac{B_1}{H'_1 L'_1} (z'_2 M'_1 + \xi L'_1) \\
 & + \frac{B_1}{H'_2 L'_2} (z'_2 M'_2 + \xi L'_2) + 2z_2 \frac{f_{33}}{\zeta} + 2z'_2 \frac{f_{33}}{\xi} \quad (3.31)
 \end{aligned}$$

Moreover

$$\begin{aligned}
 f_{13} = & \frac{f_{11} f_{33}}{\zeta} + \frac{1}{H_1} \left[ -\frac{f_{11}}{\zeta} \sum_{i=0}^{k-2} z_{i+1} R^i + f_{51, z_1} A_1 \right] \\
 & + \frac{1}{H_1} \left[ \frac{f_{51, z_1} f_{11}}{\zeta} (f_{43} f_{21} - \zeta f_{23}) + f_{23} f_{61} f_{51, z_1} \right. \\
 & \left. - f_{21} f_{63} f_{51, z_1} \right] \quad (3.32)
 \end{aligned}$$

$$\begin{aligned}
 f_{53} = & \frac{f_{51} f_{33}}{\zeta} + \frac{1}{H_1} \left[ -\frac{f_{51}}{\zeta} \sum_{i=0}^{k-2} z_{i+1} R^i + f_{11, z_1} A_1 \right] \\
 & + \frac{1}{H_1} \left[ \frac{f_{51, z_1} f_{51}}{\zeta} (f_{43} f_{21} - \zeta f_{23}) + f_{23} f_{61} f_{11, z_1} \right. \\
 & \left. - f_{21} f_{63} f_{11, z_1} \right] \quad (3.33)
 \end{aligned}$$

$$f_{23} = \frac{f_{21} f_{33}}{\zeta} + \frac{1}{H_2} \left[ -\frac{f_{21}}{\zeta} \sum_{i=0}^{k-2} z_{i+1} R^i + f_{61, z_1} A_1 \right]$$

$$\begin{aligned}
 & + \frac{1}{H_2} \left[ \frac{f_{61, z_1} f_{21}}{\zeta} (\zeta f_{13} - f_{11} f_{43}) + f_{51} f_{61, z_1} \right. \\
 & \left. - f_{11} f_{53} f_{61, z_1} \right] \quad (3.34)
 \end{aligned}$$

$$\begin{aligned}
 f_{63} = & \frac{f_{61} f_{33}}{\zeta} + \frac{1}{H_2} \left[ -\frac{f_{61}}{\zeta} \sum_{i=0}^{k-2} z_{i+1} R^i + f_{21, z_1} A_1 \right] \\
 & + \frac{1}{H_2} \left[ \frac{f_{61, z_1} f_{61}}{\zeta} (\zeta f_{13} - f_{11} f_{43}) + f_{13} f_{51} f_{21, z_1} \right. \\
 & \left. - f_{11} f_{53} f_{21, z_1} \right] \quad (3.35)
 \end{aligned}$$

**Remark :** It is noted that by similar construction, one may obtain

$$\begin{aligned}
 f_{13} = & \frac{f_{12} f_{33}}{\xi} + \frac{1}{H'_1} \left[ -\frac{f_{12}}{\xi} \sum_{i=0}^{k'-2} z'_{i+1} R^{i'} + f_{52, z_1} B_1 \right] \\
 & + \frac{1}{H'_1} \left[ \frac{f_{52, z_1} f_{12}}{\xi} (f_{43} f_{22} - \xi f_{23}) + f_{23} f_{62} f_{52, z_1} \right. \\
 & \left. - f_{22} f_{63} f_{52, z_1} \right] \quad (3.36)
 \end{aligned}$$

$$\begin{aligned}
 f_{53} = & \frac{f_{52} f_{33}}{\xi} + \frac{1}{H'_1} \left[ -\frac{f_{52}}{\xi} \sum_{i=0}^{k'-2} z'_{i+1} R^{i'} + f_{12, z_1} B_1 \right] \\
 & + \frac{1}{H'_1} \left[ \frac{f_{52, z_1} f_{52}}{\xi} (f_{43} f_{22} - \xi f_{23}) + f_{23} f_{62} f_{12, z_1} \right. \\
 & \left. - f_{22} f_{63} f_{12, z_1} \right] \quad (3.37)
 \end{aligned}$$

$$\begin{aligned}
 f_{23} = & \frac{f_{22} f_{33}}{\xi} + \frac{1}{H'_2} \left[ -\frac{f_{22}}{\xi} \sum_{i=0}^{k'-2} z'_{i+1} R^{i'} + f_{62, z_1} B_1 \right] \\
 & + \frac{1}{H'_2} \left[ \frac{f_{62, z_1} f_{22}}{\xi} (\xi f_{13} - f_{12} f_{43}) + f_{13} f_{52} f_{62, z_1} \right. \\
 & \left. - f_{12} f_{53} f_{62, z_1} \right] \quad (3.38)
 \end{aligned}$$

$$\begin{aligned}
 f_{63} = & \frac{f_{62} f_{33}}{\xi} + \frac{1}{H'_2} \left[ -\frac{f_{62}}{\xi} \sum_{i=0}^{k'-2} z'_{i+1} R^{i'} + f_{22, z_1} B_1 \right] \\
 & + \frac{1}{H'_2} \left[ \frac{f_{62, z_1} f_{62}}{\xi} (\xi f_{13} - f_{12} f_{43}) + f_{13} f_{52} f_{22, z_1} \right. \\
 & \left. - f_{12} f_{53} f_{22, z_1} \right] \quad (3.39)
 \end{aligned}$$

Thus we have the following result:

**Corollary 3.2**

If  $f_{\alpha i}$  are as in theorem(3.1) then following equations hold: R.H.S of eqn(3.32)=R.H.S of(3.36), R.H.S of eqn(3.33)=R.H.S of (3.37), R.H.S of eqn (3.34) = R.H.S of(3.38), R.H.S of eqn(3.35) =R.H.S of (3.39).

**Proof of Theorem (3.1)**

Suppose that equation(3.1) describes as  $(\zeta, \xi)$  3-dimensional P.S.P in  $R^5$  then it follows from lemma(3.1) that equations(3.5) and equations(3.6)-(3.23) are satisfied where  $f_{11, z_1}^2 + f_{21, z_1}^2 + f_{51, z_1}^2 + f_{61, z_1}^2 \neq 0$  therefore equations(3.6)-(3.23) are equivalent to the following

$$\sum_{i=0}^{k-1} z_{i+1} (f_{13,z_i} f_{51,z_1} - f_{53,z_i} f_{11,z_1}) + \zeta (f_{53} f_{51,z_1} - f_{13} f_{11,z_1}) + f_{33} H_1 + f_{51,z_1} (f_{43} f_{21} - \zeta f_{23}) = 0 \quad (3.40)$$

$$\sum_{i=0}^{k-1} z_{i+1} (f_{23,z_i} f_{61,z_1} - f_{63,z_i} f_{21,z_1}) + \zeta (f_{63} f_{61,z_1} - f_{23} f_{21,z_1}) + f_{33} H_2 + f_{61,z_1} (\zeta f_{13} - f_{11} f_{43}) = 0 \quad (3.41)$$

$$\sum_{i=0}^{k'-1} z_{i'+1} (f_{13,z_i'} f_{52,z_1} - f_{53,z_i'} f_{12,z_1}) + \xi (f_{53} f_{52,z_1} - f_{13} f_{12,z_1}) + f_{33} H_1' + f_{52,z_1} (f_{43} f_{22} - \zeta f_{23}) = 0 \quad (3.42)$$

$$\sum_{i=0}^{k'-1} z_{i'+1} (f_{23,z_i'} f_{62,z_1} - f_{63,z_i'} f_{22,z_1}) + \xi (f_{63} f_{62,z_1} - f_{23} f_{22,z_1}) + f_{33} H_2' + f_{62,z_1} (\xi f_{13} - f_{12} f_{43}) = 0 \quad (3.43)$$

$$A_1 - f_{11} f_{53} + f_{13} f_{51} - f_{21} f_{63} + f_{23} f_{61} = 0 \quad (3.44)$$

$$A_2 - f_{11} f_{23} + f_{13} f_{21} = 0 \quad (3.45)$$

$$B_1 - f_{12} f_{53} + f_{13} f_{52} - f_{22} f_{63} + f_{23} f_{62} = 0 \quad (3.46)$$

$$B_2 - f_{12} f_{23} + f_{13} f_{22} = 0 \quad (3.47)$$

$$\psi L_1 + \sum_{i=0}^{k-1} z_{i+1} (f_{13,z_i} f_{51} - f_{53,z_i} f_{11}) + \zeta (f_{53} f_{51} - f_{13} f_{11}) - f_{33} (f_{51}^2 - f_{11}^2) + f_{51} (f_{43} f_{21} - \zeta f_{23}) = 0 \quad (3.48)$$

$$\psi L_2 + \sum_{i=0}^{k-1} z_{i+1} (f_{23,z_i} f_{61} - f_{63,z_i} f_{21}) + \zeta (f_{63} f_{61} - f_{23} f_{21}) - f_{33} (f_{61}^2 - f_{21}^2) + f_{61} (\zeta f_{13} - f_{11} f_{43}) = 0 \quad (3.49)$$

$$\psi L_1' + \sum_{i=0}^{k'-1} z_{i'+1} (f_{13,z_i'} f_{52} - f_{53,z_i'} f_{12}) + \xi (f_{53} f_{52} - f_{13} f_{12}) - f_{33} (f_{52}^2 - f_{12}^2) + f_{52} (f_{43} f_{22} - \xi f_{23}) = 0 \quad (3.50)$$

$$\psi L_2' + \sum_{i=0}^{k'-1} z_{i'+1} (f_{23,z_i'} f_{62} - f_{63,z_i'} f_{22}) + \xi (f_{63} f_{62} - f_{23} f_{22}) - f_{33} (f_{62}^2 - f_{22}^2) + f_{62} (\xi f_{13} - f_{12} f_{43}) = 0 \quad (3.51)$$

Now, taking the  $z_k$  derivative of equations (3.40),(3.41) and take the  $z_{k'}$ , derivative of equations (3.42), (3.43) and also taking the  $z_{k-1}$  derivative of equation (3.44) and  $z_{k'-1}$  derivative of equations (3.45) then by using equations (3.5) we can obtain

$$f_{13,z_{k-1}} f_{51,z_1} - f_{53,z_{k-1}} f_{11,z_1} = -R^{k-1} = 0 \quad (3.52)$$

$$f_{23,z_{k-1}} f_{61,z_1} - f_{63,z_{k-1}} f_{21,z_1} = -R^{k-1} = 0 \quad (3.53)$$

$$f_{13,z_{k'-1}} f_{52,z_1} - f_{53,z_{k'-1}} f_{12,z_1} = -R^{k'-1} = 0 \quad (3.54)$$

$$f_{23,z_{k'-1}} f_{62,z_1} - f_{63,z_{k'-1}} f_{22,z_1} = -R^{k'-1} = 0 \quad (3.55)$$

$$f_{13,z_{k-1}} f_{51} - f_{53,z_{k-1}} f_{11} + (A_1)_{z_{k-1}} + f_{23,z_{k-1}} f_{61} - f_{63,z_{k-1}} f_{21} = 0 \quad (3.56)$$

$$f_{13,z_{k'-1}} f_{52} - f_{53,z_{k'-1}} f_{12} + (B_1)_{z_{k'-1}} + f_{23,z_{k'-1}} f_{62} - f_{63,z_{k'-1}} f_{22} = 0 \quad (3.57)$$

Therefore

$$f_{13,z_{k-1}} = \frac{f_{11,z_1}}{L_1} [(A_1)_{z_{k-1}} + f_{23,z_{k-1}} f_{61} - f_{63,z_{k-1}} f_{21}] \quad (3.58)$$

$$f_{53,z_{k-1}} = \frac{f_{51,z_1}}{L_1} [(A_1)_{z_{k-1}} - f_{63,z_{k-1}} f_{21} + f_{23,z_{k-1}} f_{61}] \quad (3.59)$$

$$f_{23,z_{k-1}} = \frac{f_{21,z_1}}{L_2} [(A_1)_{z_{k-1}} + f_{13,z_{k-1}} f_{51} - f_{53,z_{k-1}} f_{11}] \quad (3.60)$$

$$f_{63,z_{k-1}} = \frac{f_{61,z_1}}{L_2} [(A_1)_{z_{k-1}} + f_{13,z_{k-1}} f_{51} - f_{53,z_{k-1}} f_{11}] \quad (3.61)$$

$$f_{13,z_{k'-1}} = \frac{f_{12,z_1}}{L_1} [(B_1)_{z_{k'-1}} + f_{23,z_{k'-1}} f_{62} - f_{63,z_{k'-1}} f_{22}] \quad (3.62)$$

$$f_{53,z_{k'-1}} = \frac{f_{52,z_1}}{L_1} [(B_1)_{z_{k'-1}} + f_{23,z_{k'-1}} f_{62} - f_{63,z_{k'-1}} f_{22}] \quad (3.63)$$

$$f_{23,z_{k'-1}} = \frac{f_{22,z_1}}{L_2} [(B_1)_{z_{k'-1}} + f_{13,z_{k'-1}} f_{52} - f_{53,z_{k'-1}} f_{12}] \quad (3.64)$$

$$f_{63,z_{k'-1}} = \frac{f_{62,z_1}}{L_2} [(B_1)_{z_{k'-1}} + f_{13,z_{k'-1}} f_{52} - f_{53,z_{k'-1}} f_{12}] \quad (3.65)$$

Taking the  $z_{j+1}$  derivative of equations(3.40),(3.41) and take the  $z_{j'+1}$ , derivative of equations(3.42),(3.43) moreover taking the  $z_j$  derivative of equation(3.44) and  $z_{j'}$  derivative of equations(3.46) with  $z \leq j \leq k-1$  And  $z \leq j' \leq k'-1$  we can obtain

$$f_{13,z_j} = -\frac{1}{L_1} [f_{11} R^j - f_{11,z_1} (A_1)_{z_j} - f_{23,z_j} f_{61} f_{11,z_1} + f_{63,z_j} f_{21} f_{11,z_1}]$$

$$f_{53,z_j} = -\frac{1}{L_1} [f_{51} R^j - f_{51,z_1} (A_1)_{z_j} + f_{63,z_j} f_{21} f_{51,z_1} - f_{23,z_j} f_{61} f_{51,z_1}]$$

$$f_{23,z_j} = -\frac{1}{L_2} [f_{21} R^j - f_{21,z_1} (A_1)_{z_j} - f_{13,z_j} f_{51} f_{21,z_1} + f_{53,z_j} f_{11} f_{21,z_1}]$$

$$f_{63,z_j} = -\frac{1}{L_2} [f_{61} R^j - f_{61,z_1} (A_1)_{z_j} - f_{13,z_j} f_{51} f_{61,z_1} + f_{53,z_j} f_{11} f_{61,z_1}]$$

$$f_{13,z_{j'}} = -\frac{1}{L_1} [f_{12} R^{j'} - f_{12,z_1} (B_1)_{z_{j'}} - f_{23,z_{j'}} f_{62} f_{12,z_1} + f_{63,z_{j'}} f_{22} f_{12,z_1}]$$

$$f_{53,z_{j'}} = -\frac{1}{L_1} [f_{52} R^{j'} - f_{52,z_1} (B_1)_{z_{j'}} - f_{23,z_{j'}} f_{62} f_{52,z_1} + f_{63,z_{j'}} f_{22} f_{52,z_1}]$$

$$f_{23,z_{j'}} = -\frac{1}{L_2} [f_{22} R^{j'} - f_{22,z_1} (B_1)_{z_{j'}} - f_{13,z_{j'}} f_{52} f_{22,z_1} + f_{53,z_{j'}} f_{12} f_{22,z_1}]$$

$$f_{63,z_{j'}} = -\frac{1}{L_2} [f_{62} R^{j'} - f_{62,z_1} (B_1)_{z_{j'}} - f_{13,z_{j'}} f_{52} f_{62,z_1} + f_{53,z_{j'}} f_{12} f_{62,z_1}]$$

Also, taking the  $z_2$  derivative of (3.40), (3.41) and we get the  $z_2'$  derivative of (3.42), (3.43)

$$f_{13,z_1} f_{51,z_1} - f_{53,z_1} f_{11,z_1} = -R^1 \quad (3.66)$$

$$f_{23,z_1} f_{61,z_1} - f_{63,z_1} f_{21,z_1} = -R^1 \quad (3.67)$$

$$f_{13,z_1}' f_{52,z_1} - f_{53,z_1}' f_{12,z_1} = -R^1 \quad (3.68)$$

$$f_{23,z_1}' f_{62,z_1} - f_{63,z_1}' f_{22,z_1} = -R^1 \quad (3.69)$$

From equations(3.40)-(3.43) we can get by using equations (3.52- 3.55)and(3.66- 3.69)the following equations

$$f_{53}f_{51,z_1} - f_{13}f_{11,z_1} - \frac{1}{\zeta} \sum_{i=0}^{k-2} z_{i+1}R^i + \frac{f_{23}}{\zeta}H_1 + \frac{f_{51,z_1}}{\zeta} (f_{43}f_{21} - \zeta f_{23}) = 0 \quad (3.70)$$

$$f_{63}f_{61,z_1} - f_{23}f_{21,z_1} - \frac{1}{\zeta} \sum_{i=0}^{k-2} z_{i+1}R^i + \frac{f_{33}}{\zeta}H_2 + \frac{f_{61,z_1}}{\zeta} (\zeta f_{13} - f_{11}f_{43}) = 0 \quad (3.71)$$

$$f_{53}f_{52,z_1} - f_{13}f_{12,z_1} - \frac{1}{\xi} \sum_{i=0}^{k'-2} z_{i'+1}R^{i'} + \frac{f_{33}}{\xi}H_1' + \frac{f_{52,z_1}}{\xi} (f_{43}f_{22} - \xi f_{23}) = 0 \quad (3.72)$$

$$f_{63}f_{62,z_1} - f_{23}f_{22,z_1} - \frac{1}{\xi} \sum_{i=0}^{k'-2} z_{i'+1}R^{i'} + \frac{f_{33}}{\xi}H_2' + \frac{f_{62,z_1}}{\xi} (\xi f_{13} - f_{12}f_{43}) = 0 \quad (3.73)$$

Now from equations(3.44)and(3.45)andequations(3.70)-(3.73)one can obtain the functions  $f_{13}, f_{23}, f_{53}$  and  $f_{63}$  as given in theorem (3.1)

Moreover, from equations(3.81 – 3.84), (3.47) – (3.54)it follows that  $\psi$  is given by (3.34)Conversely, if  $f_{13}, f_{23}, f_{53}$  and  $f_{63}$  are given by (3.35 – 3.38), it follows by straight forward computation that the 1 –forms  $\omega_{\alpha 1} = f_{\alpha 1}dx + f_{\alpha 2}dy + f_{\alpha 3}dt, 1 \leq \alpha \leq 3$  satisfy the structure equations of an  $(\zeta, \xi)$  3-dim. P.S.P if  $z_{1,t} = \psi(z_0, z_1, z_1', \dots, z_k, z_k')$ .

This completes proof of theorem.It is worth mentioning that other types of evolution equations in higher dimensions shall be considered in other papers as well as the associated Backlund

transformation equations, conservation laws and the associated linear systems.

#### IV.Conclusion

In this paper, we extended the notion of P.S.P to higher dimensions i.e. 3-dim plane of constant sectional curvature-1 imbedded in  $R^5$ and we studied the change in the results and properties.

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