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On Generalized b-Closed Sets In Bitopological Spaces

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Abstract: M.Ganster & M.Steiner introduced and studied the concept of gb-closed set for general topology which motivates to introduce gb- closed sets in bitopological spaces, symbolized by ij– gb-closed sets, being the main object of this paper. This paper contains the properties of the above sets and introduces two new bitopological spaces ij - T^* $b_{1/2}$ and ij - $^*T$ $b_{1/2}$ spaces as applications.Also the relations of these spaces with ij-$T_{1/2}$ and ij-T-$b_{1/2}$ spaces have been found a place in this paper.

Key words - ij – gb-closed sets, ij - T^* $b_{1/2}$ spaces and ij - $^*T$ $b_{1/2}$ spaces.

I. Introduction & Preliminaries

A triple(X,τ_1, τ_2) , where X is a none empty set and τ_1 & τ_2 are topologies on X, is defined to be a bitopological space by J.C. Kelley[5] through the mathematical paper entilled as ‘ Bitopological spaces’ and published in proceedings of London Mathematical Society, 13 (71-89), 1963. Such a space, equipped with two arbitrary topologies, is beyond any doubt an original & fundamental work.

In 1970, g-closed sets were introduced and studied by N.Levine[6] & 1982 is the year for projection & extensive research of the concept of pre-open sets alongwith pre-continuous& research of the concept of pre-open sets alongwith pre-continuous mappings in general topology by A.S.Mashhour et.al[7] . C.Kuratowaski mentioned that a set in a space is said to be regular open set or an open domain if it is the interior of its closure.

In 1985, T.Fukutake[4] introduced and investigated the concept of g-closed sets for bitopogical spaces through the mathematical paper “ on generalized closed sets in bitopological spaces” published in Bull. FuKuoka Univ. Ed. Part III, 35(19-28), 1986.After that several authors turned their attention towards generalizations of various concepts of topology for bitopological spaces. The concepts of pre-open sets and regular open sets have been extended to bitopological spaces known as ij-pre-open and ij-regular open sets respectively as mentioned in the mathematical paper ‘on decomposition of pairwise continuity ’ by S. Sampath Kumar[10] , published in Bull. Cal. Math. Soc. 89 (1997), pp-441-446.

Andrijevic[1] introduced a new class of generalized open sets in a topological space, the so called b-open sets, which is contained in the class of semi-pre-open sets and contains all semi-open sets and pre-open sets. Also, such a class of b-open sets generates the same topology as the class of pre-open sets. Extensive research on generalizing of closedness was done in the recent years and in 2007, M. Ganster & M.steneier presented the concept & the study of generalized b-closed(briefly , gb-closed) sets in topology. Also in 2009, generalized b-closed sets were analysed in a fundamental way by Al-Omari et. al [2].

The aim of this paper is to extened the concept of gb-closed sets in general topology for bitopology , called ij- gb-closed sets. properties of these sets are investigated and two new bitopological spaces ij - T^* $b_{1/2}$ and ij - $^*T$ $b_{1/2}$ spaces are introduced as applications.

All through this paper, the spaces X and Y, (or X,τ_1, τ_2) and (Y,σ_1,σ_2) stand for bitopological spaces with no separation axioms, unless otherwise stated.

The interior (resp. closure) of a subset A of a bitopogical space (X,τ_1, τ_2) w.r.t. topology τ_i (i = 1,2) is denoted by τ_i –int(A)(resp. τ_i- cl(A)). As usual the set of all closed(resp. open) sets w.r.t. topology τ_i is denoted by i-C(X)(resp. i- O(X)).

The following definitions being useful in the sequel, are called:

Definition (1.1): A subset A of a bitopological space (X,τ_1, τ_2) is said to be

(a) ij-pre-open (briefly, ij-p-open)[4] A $\subseteq$ τ_i -int (τ_j –cl(A)),
(b) ij-regular-open (briefly, ij-r-open) [4]A = τ_i -int (τ_j –cl(A)),
(c) ij-semi-open (briefly, ij-s-open)[9] A $\subseteq$ τ_i -cl (τ_j –int(A)),
(d) ij-regular-closed (briefly, ij-r-closed)[9] A = τ_i -cl (τ_j –int(A)),
(e) ij-generalized closed (briefly, ij-g-closed)[4] A $\subseteq$U & U $\in$ τ_i $\Rightarrow$ τ_j–cl(A)$\subseteq$ U,
(f) ij-regular generalized closed (briefly, ij-rg-closed)[3] A $\subseteq$U & U $\in$ ij-RO(X) $\Rightarrow$ τ_j–cl(A)$\subseteq$ U,

Naturally, the complement of respective open/closed set is respective closed/open set .

The class of ij-$\mathcal{K}$-open (resp. ij- $\mathcal{K}$-closed) sets is denoted by the symbol ij-$\mathcal{L}$O(X)(resp. ij-$\mathcal{L}$C(X)) where $\mathcal{K}$ = p, regular(r), s,g,rg & $\mathcal{L}$ = P,R,S,G,RG accordingly.

Definition (1.2): Andrijevic mentioned the following characteristic [prop. 2.1[1]] for a set A of a topological space (X,T) to be a b-open set using preclosure & pre interior operators as :

A subset A of a space (X, τ) is b-open if and only if A $\subseteq$ pcl(pint(A)).

The above definition motivates the author to coin the the concept of an ij-b-open set in a bitopological space (X,τ_1, τ_2) in the existence sense expressed through the following manner:

Definition (1.3): A subset A of a bitopological space(X,τ_1, τ_2) is said to be

(a) ij-b-open set if A $\subseteq$ τ_i -pcl (τ_j –pint(A)),
(b) ij-b-closed set if τ_i -pint (τ_j –pcl(A)) $\subseteq$ A or A^c ij-b-open.

Symbols ij-BO(X) & ij-BC(X) stand for the class of all ij-b-open & ij-b-closed sets respectively.

Definition (1.4): For any bitopological space (X,τ_1,τ_2) and A ⊆ X , ij-b-interior & ij-b-closure of A are denoted by ij-bint(A) & ij-bcl(A) respt. & defined as:

ij-bint(A) = ∪ { F⊆ : F∈ ij-BO(X), F⊆ A}

& ij-bcl(A) = ∩ { F⊆ X : F∈ ij-BC(X), A⊆ F}

II. Generalized b-Closed Sets i.e. ij-gb-Closed Sets And ij-gbr- Closed Sets

In this section, the concepts of ij-gb-closed & ij-gbr –closed sets in bitopological spaces are introduced and discussed the related properties.

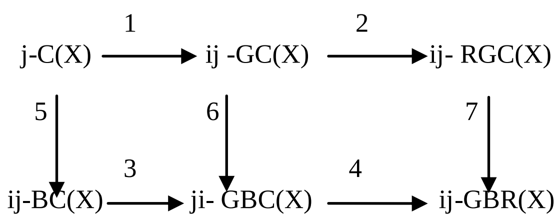
Definition (2.1): A subset A of a bitopological space(X,τ_1,τ_2) is said to be

(a) ij-generalized b-closed (briefly ij-gb-closed) set if A ⊆U & U ∈ τ_i ⇒ ji-bcl(A)⊆ U,

(b) ij-generalized b-regular closed (briefly ij-gbr-closed) set if A ⊆U & U ∈ ij-RO(X) ⇒ji -bcl(A)⊆ U.

As usual ij-GBC(X) & ij- GBRC(X) represent the class of all ij-gb-closed and ij-gbr –closed sets respectively, for a bitopological space (X,τ_1,τ_2).

The following diagram shows the relationship between above different types of closed sets:



None of these implication is reversible.

Example (2.3): Suppose that X = {a,b,c,d} , τ_1 = {X,φ,{a},{a,d}} and τ_2 = {X,φ, {a,b},{c,d}}.

Now, (Arrows 1,5) {b} ∈ 12 – GC(X) ∩ 12- BC(X) but {b} ∉ 2 – C(X).

(Arrows 2,6) {d} ∈ 12 RGC(X) ∩ 21- GBC(X) but {d} ∉ 12 – GC(X),

Since, there exists {a,d} ∈ τ_1 containing {d} such that 2-cl({d}) ={c,d} ⊈ {a,d}.

(Arrow 3) {b,c} ∈ 21 – GBC(X) but {b,c} ∉ 12 – BC(X).

(Arrow 4) {a,b}∈ 12 – GBRC(X) but {a,b} ∉ 12 – GBC(X).

(Arrow 7) {c} ∈ 21 – GBRC (X) but {c}∉ 21- RGC(X), since, there exists {c,d} ∈ 21- RO (X) containing {c} such that 1-cl ({c,d}) = {c,d} ⊄ {c}.

Example (2.4): As in example (2.3), we observe that

(a) {c,d} ∈ 21-RGC(X) but {c,d}∉12 – GBC(X) .Also {c} ∈ 12 – GBC (X) but {c}∉ 21- RGC(X).

(b) {a,c} ∈ 12-GC(X) but {a,c}∉12 – BC(X). Also, {d} ∈ 12 – BC (X) but {d}∉ 12- GC(X).

Thus, the classes ij-GC(X) and ij-BC(X) are independent and also, the classes ij-RGC(X) and ji-GBC(X) are independent.

Theorem (2.5): If A is an ij-gb-closed set, then { ji-bcl(A) – A} does not contain any nonempty T_i-closed set.

Proof: let F ∈ i-C(X) such that F ⊆ ji-bcl(A)-A. Since, F^c is T_i-open, A ⊆F^c and A is ij-gb-closed, it follows that ji-bcl(A) ⊆F^c. This means that F⊆ $[\text{ji-bcl(A)}]^c$.

Hence, F⊆ $\{\text{ji-bcl(A)}\}^c$ ∩{ji-bcl(A) – A}

i.e. F⊆ $\{\text{ji-bcl(A)}\}^c$ ∩{ji-bcl(A) ∩ A^c} = φ.

i.e. F = φ.

Theorem (2.6): An ij-gb-closed subset A of a bitopological space (X,τ_1,τ_2) is ji-b-closed iff {ji-bcl(A) – A }is i-closed.

Proof : Assume that A is an ij-gb-closed subset of a bitopological space (X,τ_1,τ_2) such that it is also ji-b-closed.

i.e. ji-bcl(A) = A and consequently ji-bcl(A)- A = φ which belongs to i-C(X).

Conversely, let {ji-bcl(A)-A} be i-closed. Then by theorem (2.5) , {ji-bcl(A)-A} does not contain any nonempty τ_i- closed subset and since,{ji-bcl(A)-A} is τ_i-closed subset of itself , hence, {ji-bcl(A)-A} =φ. This means that A = {ji-bcl(A)}. i.e. A is ji-b-closed.

Definition (2.7): A point p ∈X is said to be an ij-b-limit point of a subset A of a bitopological space if for each ij-b-open set G containing p, we have G∩{A-{p}}≠ φ.

The set of all ij-b-limit points of A is called ij-b-derived set of A and is denoted by ij-D_b(A).

Since, every j-open set is ij-b-open, hence, ij-D_b(A) ⊆ j-D(A) where A ⊆ X , j-D(A) is the j-derived set of A. Moreover, since every j-closed set is ij-b-closed, hence, A ⊆ ij-bcl(A) ⊆ j-cl(A) & A ⊆ ji-bcl(A) ⊆ i-cl(A)

Lemma (2.8): If j-D(A) = ij-D_b(A), & i-D(A) = ji-D_b(A), then j-cl(A) = ij-bcl(A); & i-cl(A) = ji-bcl(A) respectively where A⊆ X.

Lemma (2.9): If j-D(F) ⊆ ij-D_b(F), for every subset F of (X,τ_1,τ_2) , then ij-bcl(A ∪ B) = ij-bcl(A) ∪ ij –bcl(B), where A,B ⊆ X.

Theorem (2.10): If A and B are ij-gb-closed sets such that i-D(A) ⊆ ji-D_b(A) & i-D(B) ⊆ ji-D_b(B) , then A ∪ B is ij-gb-closed.

Proof : Let U be an i-open set such that A ∪ B ⊆ U where (X,τ_1,τ_2) is a bitopological space. Since, A and B are ij-gb-closed sets, hence ji-bcl(A) ⊆ U & ji-bcl(B) ⊆ U.

Since, i-D(A) ⊆ ji-D_b(A), hence, i-D(A)= ji-D_b(A) and by lemma (2.8), i-cl(A) = ji-bcl(A). Similarly, i-cl(B) = ji-bcl(B).

Thus, ji-bcl(A∪ B) ⊆ i-cl(A∪B) = i-cl(A) ∪ i-cl(B) = ji-bcl(A) ∪ ji –bcl(B) ⊆ U, which implies that A∪ B is ij-gb-closed.

Definition (2.11): In a bitopological space (X,τ_1,τ_2) , let B ⊆A ⊆X & (A,τ_{1A},τ_{2A}) , be induced bitopological subspace.

Then the set B is known to be ij-gb-closed relative to A if ji-bcl_A(B) ⊆ U where B ⊆ U and U is i-open in A.

Theorem (2.12): for an ij-gb- closed and T_i-open set A in a bitopological space (X,τ_1,τ_2), the set B ⊆A is ij-gb-closed relative to A iff B is ij-gb-closed in X.

Proof : Since, A is both ij-gb-closed and τ_i-open set in a bitopological space, hence, ji-bcl(A) ⊆ A. Also, B⊆A provides that ji-bcl(B) ⊆ji-bcl(A). Combining these facts, we have ji-bcl(B) ⊆ji-bcl(A) ⊆A.

Now, A ∩ ji-bcl(B) = ji-bcl_A(B). Using it, we get ji-bcl_A(B) = ji-bcl(B) ⊆A. If B is ij-gb-closed relative to A and U is τ_i-open set in X such that B ⊆U, then B = B∩A ⊆ U ∩A where U∩A is τ_{iA}-open(or i-open in A).

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Hence as B is ij-gb-closed relative to A, ji-bcl(B) = ji-bcl_A(B) ⊆ U ∩ A ⊆U. Consequently, B is ij-gb-closed in X.

Conversely, if B is – ij-gb-closed in X and U is an i-open subset of A such that B⊆U, then U = V∩A for some i-open subset V of X. As B⊆V and B is ij-gb-closed set in X, ji-bcl(B) ⊆ V.

Thus, ji—bcl_A(B) = ji-bcl(B) ∩ A ⊆ V∩ A = U. Consequently B is ij-gb-closed relative to A.

Corollary (2.13): If A is an ij-gb-closed & τ_i-open set in a bitopological space(X,τ_1,τ_2) then A ∩ F is also ij-gb-closed whenever F ∈ ji-BC(X).

Proof : Let A be an ij-gb-closed & τ_i-open set in a bitopological space(X,τ_1,τ_2). For A to be ij-gb-closed as well as T_i –open , it is natural that ji-bcl(A) ⊆ A. So, A is ji-bclosed.

Again, as F ∈ ji-BC(X) & A ∈ ji-BC(X) so A∩ F ∈ ji-BC(X). Now, A∩ F ⊆ A ⟹ ji-bcl(A∩ F) ⊆ A which means that A ∩ F is ij-gb-closed.

Theorem (2.14): If A is an ij-gb-closed set and B is any set such that A ⊆B⊆ji-bcl(A), then B is also an ij-gb-closed set.

Proof : Let B ⊆U where U is τ_i-open in (X,τ_1,τ_2). Since, A is ij-gb-closed and A ⊆U, then ji-bcl(A) ⊆U.

Again, A ⊆B⊆ji-bcl(A) ⟹ ji-bcl(A) = ji-bcl(B). Therefore, combining these facts, we conclude that ji-bcl(B) ⊆U whenever B ⊆U & U is τ_i-open . So, B is also an ij-gb-closed set.

Theorem (2.15): A subset A of a bitopological space (X,τ_1,τ_2) is ij-gb-open iff F ⊆ji-bint(A) whenever F is τ_i -closed and F ⊆A.

Proof : Let A be an ij-gb-open set in (X,τ_1,τ_2) and F ⊆A where F is τ_i –closed. Then A^C is an ij-gb-closed set contained in the τ_i-open set F^C. So, ji-bcl(A^C) ⊆F^C. This implies that $\{$ji-bint(A)$\}^C$ ⊆F^C and consequently, F ⊆ ji-bint(A).

Conversely, if F is a τ_i -closed set with F ⊆ji-bint(A) and F ⊆A, then $\{$ji-bint(A)$\}^C$ ⊆F^C. Thus, ji-bcl(A^C) ⊆F^C. Consequently, A^C is an ij-gb-closed set and A is an ij-gb-open set.

III. ij - $T^* b_{1/2}$ and ij - $^*T b_{1/2}$ spaces

This section introduces ij - $T^* b_{1/2}$ and ij - $^*T b_{1/2}$ bitopological spaces along with ij - T $b_{1/2}$ spaces.

Definition (3.1): A bitopological space (X,τ_1,τ_2) is said to be an ij - T $b_{1/2}$ space if every ji-gb-closed set is an ij-b-closed set.

Definition (3.2): A bitopological space (X,τ_1,τ_2) is said to be an ij - $T_{1/2}$ space if every ij-g-closed set is a j-closed set.

Definition (3.3): A bitopological space (X,τ_1,τ_2) is said to be an ij - $T^* b_{1/2}$ space if every ji-gb-closed set is j-closed .

Definition (3.4): A bitopological space (X,τ_1,τ_2) is said to be an ij - $^*T b_{1/2}$ space if every ji-gb-closed set is ij-g-closed .

Proposition (3.5): For each element x of (X,τ_1,τ_2),{x} is either of the following facts :

(a) i-closed or ij-gb-open. (b) j-closed or ji-gb-open.

Proof : Let x ∈X where (X,τ_1,τ_2) is bitopological space.

(a) Suppose that {x} is not τ i-closed, then X is the only τ_i-open set such that $\{x\}^C$⊆ X. This means that ji-bcl($\{x\}^C$)⊆ ji-bcl(X) = X which in turns establishes that $\{x\}^C$ is ij-gb-closed i.e. {x} is ij-gb-open.

(b) Similarly, {x} is j-closed or ji-gb-open.

Proposition (3.6): If (X,τ_1,τ_2) is an ij - $T^* b_{1/2}$ space, then it is an ij - $T_{1/2}$ space but not conversely.

Proof : Let (X,τ_1,τ_2) be an ij - $T^* b_{1/2}$ space, then every ji-gb-closed set is j-closed.

Since, the class of all ji-gb-closed sets contains the class of ij-g-closed sets i.e. ji-GBC(X) ⊇ ij-GC(X), hence, every ij-g-closed set is j-closed and consequently, (X,τ_1,τ_2) is an ij - $T_{1/2}$ space.

Obviously, the converse is not true.

Proposition (3.7): A bitopological space (X,τ_1,τ_2) is an ij - $T^* b_{1/2}$ space iff {x} is j-open or j-closed for each x ∈X.

Proof : Let x ∈X where (X,τ_1,τ_2) is a bitopological space. First, We assume that (X,τ_1,τ_2) is an ij - $T^* b_{1/2}$ space. So every ji-gb-closed set is j-closed. By proposition (3.5), let {x} be either j-closed or ji-gb-open. Let {x} , not be j- closed, then X is ji-gb-open i.e. $\{x\}^C$ is ji-gb-closed and in turns $\{x\}^C$ is j-closed i.e. {x} is-j-open. Thus{x} is either j-open or j-closed.

Conversely, let F be an ji-gb-closed set. By assumption, {x} is either j-open or j-closed for any x ∈ ij-gbcl(F).

Case I : Suppose that {x} is j-open. Since,{x} ∩ F ≠φ, we have x ∈F.

Case II: Suppose that {x} is j-closed. If x ∉F, then {x} ⊆{ij-gbcl(F)-F}, which is a contradiction to theorem (2.5). Therefore, x ∈F.

Thus, in both the cases, we conclude that F is j-closed. Hence, (X,τ_1,τ_2) is an ij - $T^* b_{1/2}$ space.

Corollary : A space ,(X,τ_1,τ_2) is an ji - $T^* b_{1/2}$ iff {x} is either i-open or i-closed for each x ∈X .

Proposition (3.8): If (X,τ_1,τ_2) is an ij - $T^* b_{1/2}$ space, then it is also ij - $^*T b_{1/2}$ space, but not conversely.

Proof : Let (X,τ_1,τ_2) be an ij - $T^* b_{1/2}$ space, then every ji-gb-closed set is j-closed. We, however, know that every j-closed set is ij-g-closed set. So, here, every ji-gb-closed set is ij-g-closed and consequently, (X,τ_1,τ_2) is ij - $^*T b_{1/2}$ space.

Obviously, the converse does not exist.

Proposition (3.9): If (X,τ_1,τ_2) is an ij - $^*T b_{1/2}$ as well as an ij - $T_{1/2}$ space, then it is an ij - $T^* b_{1/2}$ space.

Proof : Let (X,τ_1,τ_2) be an ij - $^*T b_{1/2}$ space. Then, every ji-gb-closed set is ij-g-closed.

Also, (X,τ_1,τ_2) is an ij - $T_{1/2}$ space. Then, every ij-g-closed set is j-closed.

Combining these two facts, we conclude that in space (X,τ_1,τ_2) every ji-gb-closed set is j-closed. Consequently, (X,τ_1,τ_2) is an ij - $T^* b_{1/2}$ space.

Corollary: (1) An ij - $^*T b_{1/2}$ space & ij-$T_{1/2}$ is ij-T$b_{1/2}$.

Proof : Since, ij - $^*T b_{1/2}$ & ij - $T_{1/2}$ space, is an ij - $T^* b_{1/2}$ space. Hence, every ji-gb-closed set is j-closed. And as a j-closed set is an ij-b-closed set. So, here, every ji-gb-closed set is ij-b-closed. Thus, it is an ij-T$b_{1/2}$ space.

(2) ij - $T^* b_{1/2}$ space is ij-T$b_{1/2}$ space but not conversely.

IV. Conclusion

The concepts of gb –closed sets in a bitopological space (X,τ_1,τ_2) i.e. ij-gb-Closed Sets And ij-gbr- Closed Sets have been introduced and investigated with their important properties & furthermore they have been ,here, analyzed. As applications of these concepts two new bitopological spaces ij-T*$b_{1/2}$ & ij-*T$b_{1/2}$ have been explored. The fundamental characteristics of such spaces have also been analyzed and correlated with ij-T$b_{1/2}$ space. The structural properties of these spaces have been emphasized to create the vast canvas in the world of Mathematics and have wide utility which surely pleases the mathematician if one of his abstract structures finds an application.

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